On Differentials of Functions in certain cases only *

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§337 If y was any function of x and this variable quantity is increased by ω that x goes over into $x+\omega$, the function will have this value

$$y+\frac{\omega dy}{dx}+\frac{\omega^2 ddy}{2dx^2}+\frac{\omega^3 d^3y}{6dx^3}+\frac{\omega^4 d^4y}{24dx^4}+\text{etc.}$$

and hence will receive this increment

$$\frac{\omega dy}{dx}+\frac{\omega^2 ddy}{2dx^2}+\frac{\omega^3 d^3y}{6dx^3}+\frac{\omega^4 d^4y}{24dx^4}+\text{etc.},$$

as we showed above [§ 48]. Therefore, if $\omega = dx$ such that x grows by the amount of its differential dx, the function y will receive the increment

$$=dy+\frac{1}{2}ddy+\frac{1}{6}d^3y+\frac{1}{24}d^4y+\text{etc.},$$

which will be the true differential of y. But since any arbitrary term of this series has an infinite ratio to the following, with respect to the first all vanish such that dy taken in usual manner yields the true differential of y. In like manner the true second, third, fourth etc. differentials of y will be as follows

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$$
\begin{aligned}
dd.y &= ddy + \frac{3}{3}\ d^3y + \frac{7}{3\cdot 4}d^4y + \frac{15}{3\cdot 4\cdot 5}d^5y + \frac{31}{3\cdot 4\cdot 5\cdot 6}\ d^6y\ + \text{etc.}\\
d^3.y &= d^3y + \frac{6}{4}\ d^4y + \frac{25}{4\cdot 5}d^5y + \frac{90}{4\cdot 5\cdot 6}d^6y + \frac{301}{4\cdot 5\cdot 6\cdot 7}\ d^7y\ + \text{etc.}\\
d^4.y &= d^4y + \frac{10}{5}d^5y + \frac{65}{5\cdot 6}d^6y + \frac{350}{5\cdot 6\cdot 7}d^7y + \frac{1701}{5\cdot 6\cdot 7\cdot 8}\ d^8y\ + \text{etc.}\\
d^5.y &= d^5y + \frac{15}{6}d^6y + \frac{140}{6\cdot 7}d^7y + \frac{1050}{6\cdot 7\cdot 8}d^8y + \frac{6951}{6\cdot 7\cdot 8\cdot 9}\ d^9y\ + \text{etc.}\\
d^6.y &= d^6y + \frac{21}{7}d^7y + \frac{266}{7\cdot 8}d^8y + \frac{2646}{7\cdot 8\cdot 9}d^9y + \frac{22827}{7\cdot 8\cdot 9\cdot 10}d^{10}y + \text{etc.}\\
&\text{etc.}
\end{aligned}
$$

which follow from § 56, if one writes dx instead of ω. Therefore, exactly those differentials of y will be complete, in which not even the terms vanishing with respect to the first are neglected. But these terms are found, if the function y is continuously differentiated and dx was put to be constant. So having put $y = ax - xx$, because of

$$dy = adx - 2xdx \quad \text{and} \quad ddy = -2dx^2,$$

the complete differentials of y will be

$$dy = adx - 2xdx - dx^2, \quad ddy = -2dx^2;$$

the following are all zero.

§338 Although in general in these expressions of the differentials the following terms with respect to the first are considered to be zero, nevertheless in special cases, in which the first term vanishes, this assumption is not valid and the second term can not be neglected anymore. Therefore, even though in the preceding example the differential of the formula $y = ax - xx$ in general is $= (a - 2x)dx$, having neglected the term $-dx^2$, which certainly is infinite times smaller than the first $(a - 2x)dx$, here nevertheless this condition that the first term does not vanish per se is considered to be satisfied. Therefore, if the differential of $y = ax - xx$ is in question in the case $x = \frac{1}{2}a$, one has to say that actually $= -dx^2$; if the variable x increases by the differential dx, the decrement of the function y in the case $x = \frac{1}{2}a$ will be dx^2. But, having excluded this single case, the differential of the function y will always be $= (a - 2x)dx$; for, if not $x = \frac{1}{2}a$, the second term $-dx^2$, with respect to the first, is always justly neglected. And the negligence of the term dx^2 cannot induce an error even in the case $x = \frac{1}{2}a$; for, usually the first differentials are compared to each other; hence, because $dy = -dx^2$ in the case $x = \frac{1}{2}a$ vanishes in comparison to the first differentials dx, it does not matter, whether in this case we have $dy = 0$ or $dy = -dx^2$.

§339 While y denotes any function of x, having taken the differentials several times, let

$$dy = pdx, \quad dp = qdx, \quad dq = rdx, \quad dr = sdx \quad \text{etc.}$$

Hence, the complete differentials of y in which nothing is neglected will be

$$\begin{aligned}
d.y &= pdx + \frac{1}{2}q\, dx^2 + \frac{1}{6}r\, dx^3 + \frac{1}{24}sdx^4 + \frac{1}{120}tdx^5 + \text{etc.}\\
d^2.y &= qdx^2 + r\, dx^3 + \frac{7}{12}sdx^4 + \frac{1}{4}t\, dx^5 + \text{etc.}\\
d^3.y &= rdx^3 + \frac{3}{2}s\, dx^4 + \frac{5}{4}t\, dx^5 + \text{etc.}\\
d^4.y &= sdx^4 + 2t\, dx^5 + \text{etc.}\\
d^5.y &= tdx^5 + \text{etc.}\\
&\text{etc.}
\end{aligned}$$

Therefore, if the first terms of these expressions do not vanish, they alone will exhibit the differentials of y; but, if in a certain case the first term becomes $= 0$, the following term will express the differential in question. And if the second term also vanishes, the third term will yield the value of the differential in question; but if even this term vanishes, the fourth, and so forth. Therefore, it is understood that the first differential of any function of x never vanishes completely; for, even though $p = 0$, in which case sy is usually considered to vanish, this differential will then be expressed by a higher power of dx, as, e.g., either by $\frac{1}{2}qdx^2$ or, if also $q = 0$, by $\frac{1}{6}rdx^3$, and so forth.

§340 But although in these cases the differential of y, with respect to higher first differentials it is compared to, is justly neglected and considered to be zero, it is nevertheless often helpful to also know its true expression. For, from the complete form of the differential it is immediately seen in which cases the given function has a maximum or minimum value. For, if it was

$$d.y = pdx + \frac{1}{2}qdx^2 + \frac{1}{6}rdx^3 + \text{etc.},$$

for y to have a maximum or minimum value it is necessary that $p = 0$; therefore, in this case it will be $dy = \frac{1}{2}qdx^2$ and the function y, if one writes $x \pm dx$ instead of x, goes over into $y + \frac{1}{2}qdx^2$ and will therefore have a minimum value, if q has a positive value, but maximum value, if q has a negative value. But if at the same time $q = 0$, it will be $dy = \frac{1}{6}rdx^3$ and the function will go over into $y \pm \frac{1}{6}rdx^3$ by writing $x \pm dx$ instead of x and in this case neither a maximum nor a minimum value results; but if also $r = 0$, then, having written $x \pm dx$ instead of x, the function y will become $= y + \frac{1}{24}sdx^4$, which exhibits a maximum, if s was a negative quantity, a minimum on the other hand, if s is a positive quantity. Other occasions in which the complete expression of the differentials have a use will occur below.

§341 Let us put that p vanishes in the case $x = a$, what happens, if it was $p = (x-a)P$. But such a value results, if it was

$$y = (x-a)^2P + C$$

while C denotes any constant quantity. For, because

$$pdx = (x-a)^2dP + 2(x-a)Pdx,$$

it will certainly be $p = 0$ for $x = a$. Therefore, because of

$$dpdx = qdx^2 = (x-a)^2ddP + 4(x-a)dPdx + 2Pdx^2,$$

having put $x = a$, it will be $qdx^2 = 2Pdx^2$ and the complete differential in this case $x = a$ will be

$$dy = Pdx^2,$$

if not by accident also P vanishes for $x = a$ which cases I will contemplate later.

But the present case can be exhibited more generally this way. Let

$$z = (x-a)^2 P + C$$

and let y be any function of z such that $dy = Zdz$ while Z denotes any function of $z = (x-a)^2 P + C$. Therefore, it will be

$$dz = (x-a)^2 dP + 2(x-a)Pdx \quad \text{and} \quad pdx = Z(x-a)^2 dP + 2Z(x-a)Pdx,$$

which term becomes $= 0$, if $x = a$; and in the same case, having neglected the terms containing the factor $x - a$, it will be $qdx^2 = 2PZdx^2$ and hence in the case $x = a$ it will be $dy = PZdx^2$, after in PZ it was put a for x everywhere. Hence, if y was any function of $z = (x-a)^2 P + C$ such that $dy = Zdz$, in the case $x = a$ the differential will be

$$dy = PZdx^2.$$

Therefore, this function y has maximum value in the case $x = a$, if in the same case PZ was a negative quantity, a minimum value on the other hand, if PZ was a positive quantity.

§342 If it was $p = (x-a)^2 P$, in the case $x = a$ also q vanishes; but such an expression results for p, if it was

$$y = (x-a)^3 P + C.$$

Therefore, it will be

$$pdx = (x-a)^2 dP + 3(x-a)Pdx,$$
$$qdx^2 = (x-a)^3 ddp + 6(x-a)^2 dPdx + 6(x-a)Pdx^2,$$

both sides of which vanish in the case $x = a$; but the following will be

$$rdx^3 = (x-a)^3 d^3 P + 9(x-a)^2 ddpdx + 18(x-a)dPdx^2 + 6Pdx^3 = 6Pdx^3$$

having put $x = a$. Hence, because p and q vanish in the case $x = a$, it will be

$$dy = \frac{1}{6} rdx^3 = Pdx^3.$$

In like manner, if one puts

$$z = (x-a)^3 P + C$$

and y was any function of z such that $dy = Zdz$, because of

$$dz = (x-a)^3 dP + 3(x-a)^2 P dx,$$

it will also be $p = 0$ and $q = 0$ and it will be $rdx^3 = 6PZdx^3$; hence, in the case $x = a$ it will be

$$dy = PZdx^3.$$

Therefore, this function y, even though in the case $x = a$ $p = 0$, will nevertheless have neither a maximum nor minimum value.

§343 These differentials can be found more easily from the nature of the differentials. For, since the differential of y results, if y is subtracted from the closest following state which results, if one writes $x + dx$ instead of x, let us in the first case, in which it was

$$y = (x-a)^2 P + C,$$

write $x + dx$ instead of x and it will be

$$y^{\mathrm{I}} = (x - a + dx)^2 P^{\mathrm{I}} + C,$$

whence it will be

$$dy = (x - a + dx)^2 P^{\mathrm{I}} - (x-a)P.$$

Therefore, in the case $x = a$ it will be $dy = P^{\mathrm{I}} dx^2$, and because P^{I} and P have the ratio of 1, it will be

$$dy = Pdx^2.$$

In like manner, if it was

$$z = (x-a)^2 P + C,$$

it will be $dz = Pdx^2$; hence, if y is any function of z such that $dy = Zdz$, it will be

$$dy = PZdx^2$$

in the case in which one puts $x = a$.

Further, if

$$z = (x-a)^3 P + C,$$

it will be $z^{\mathrm{I}} = (x-a+dx)^3 P^{\mathrm{I}} + C$ and therefore in the case $x = a$ it will be

$$z^{\mathrm{I}} - z = dz = Pdx^3.$$

Hence, if y was any function of z and $dy = Zdz$, the differential in the case $x = a$ will also be

$$dy = PZdx^3,$$

if in the functions P and Z one substitutes a for x everywhere. Since in this case $z = C$ and Z is a function of z, Z will become a constant quantity, such a function of C, of course, as it was one of z before.

§344 Therefore, if it was in general

$$y = (x-a)^n P + C,$$

since

$$y^{\mathrm{I}} = (x-a+dx)^n P^{\mathrm{I}} + C,$$

in the case $x = a$ it will be

$$dx = Pdx^n;$$

whence, if it was $n > 1$, this differential will vanish compared to the higher first differentials which are homogeneous to dx. Therefore, from the preceding it is obvious that the function y has a maximum or minimum value in the case $x = a$, if n was an even number; for, then, if for $x = a$ P becomes a positive quantity, y will have a minimum; but if P was a negative quantity, y will have a maximum. And this way the nature of maxima and minima is found a lot more easily than by using the method explained above, since it is not necessary to consider higher differentials. If

$$z = (x-a)^n P + C$$

and y was any function of z that $dy = Zdz$, in the case $x = a$ the differential will be

$$dy = PZdx^n.$$

But it is to be noted that here n it to be taken positively or greater than 0; for, if n was a negative number, then $(x-a)^n$ would not vanish for $x = 0$, as we assumed, but would even become infinitely large.

§345 Now, we saw that this way the differential can be found a lot more conveniently than by means of the series we used to express the complete differential before; for, if n was an integer number, so many terms of that series have to be considered as n contains unities. But if n was a fractional number, this series will not even ever exhibit the true differential. For, let us put that

$$y = (x-a)^{\frac{3}{2}} + a\sqrt{a};$$

if we consider the series

$$dy = pdx + \frac{1}{2}qdx^2 + \frac{1}{6}rdx^3 + \frac{1}{24}dx^4 + \text{etc.},$$

it will be

$$p = \frac{3}{2}\sqrt{x-a}, \quad q = \frac{3}{4\sqrt{x-a}}, \quad r = \frac{-3}{8(x-a)\sqrt{x-a}},$$

$$s = \frac{9}{16(x-a)^2\sqrt{x-a}} \quad \text{etc.}$$

Hence, if one puts $x = a$, it will be $p = 0$, but all following terms q, r, s etc. will become infinite; hence, the value of the differential dy cannot be defined in this case at all. But the method deduced from the nature of differentials leaves no doubt. For, since $y = (x-a)^{\frac{3}{2}} + a\sqrt{a}$, having written $x + dx$ instead of x, it will be $y^{\mathrm{I}} = (x - a + dx)^{\frac{3}{2}} + a\sqrt{a}$ and, if one puts $x = a$, it will be $dy = dx\sqrt{dx}$. Therefore, this differential vanishes with respect to dx; but the second differentials homogeneous to dx^2 will vanish with respect to the latter.

§346 Let us expand these cases in which the exponent n is a fractional number a little more accurately and let be

$$y = P\sqrt{x-a} + C;$$

because of $y^{\mathrm{I}} = P^{\mathrm{I}}\sqrt{x-a+dx} + C$, it will be

$$dy = P\sqrt{dx}$$

in the case $x = a$; therefore, this differential will have an infinite ratio to dx and to the differentials homogeneous to dx. Hence it is plain, what is to be said about the nature of maxima and minima in this case. For, because, having written $a + dx$ instead of x, y goes over into

$$C + P\sqrt{dx},$$

because of the ambiguous $\sqrt{dx}$, the function y will obtain two values; the one greater than C, which it receives for $x = a$, the other smaller; hence in the case $x = a$ it will have neither a maximum nor a minimum value. Furthermore, if dx is taken negatively, the value of y will even become imaginary. The same is to said, if $z = P\sqrt{x-a} + C$ and y is any function of z that it is $dy = Zdz$; for, then it will be $dy = PZ\sqrt{dx}$ in the case $x = a$.

§347 If this function was propounded

$$y = (x-a)^{\frac{m}{n}}P + C,$$

whose differential is in question in the case $x = a$, it will, as we concluded from the preceding, be

$$dy = Pdx^{\frac{m}{n}}.$$

Therefore, if it was $m > n$, this differential will vanish with respect to dx; but if $m < n$, the ratio $\frac{dy}{dx}$ will be infinitely large. Furthermore, if n is an even number, the differential will have two values, the one positive, the other negative; and so the function y which in the case $x = a$ becomes $= C$, if one puts $x = a + dx$, will have two values, the one greater than C, the other smaller; but if one would put $x = a - dx$, then y would even become imaginary; hence, in this case y would become neither a maximum nor a minimum. Now, let us put that the denominator n is an odd number; the numerator m will either be even or odd. Let m be an even number at first; since dy retains the same value, no matter whether dx is taken positively or negatively, it is perspicuous that the function y in the case $x = a$ has either a maximum or minimum value depending on whether in this case P was a negative or positive quantity. But if both numbers m and n were odd, the differential dy will go over into its negative, if dx was negative; and therefore, the function y will have neither a maximum nor a minimum in this case, if one puts $x = a$.

§348 If the function y consists of several terms of this kind, each of which is divisible by $x - a$, such that

$$y = (x-a)^m P + (x-a)^n Q + C,$$

in the case $x = a$ its differential will be

$$dy = Pdx^m + Qdx^n;$$

in this expression, if it was $n > m$, the second term vanishes in comparison to the first such that only $dy = Pdx^m$ results. But if n was a fraction with an even denominator, then, even though Qdx^n vanishes with respect to Pdx^m, it can nevertheless not be completely neglected. For, from this it is clear, if dx is taken negatively, that the value of dy becomes imaginary, whereas from the first term Pdx^m only this is not obvious. Therefore, since, if n is a fraction with an even denominator, dx cannot be taken negatively, but if it is taken positively, the term Qdx^n yields two values, the function $y = (x-a)^m P + (x-a)^n Q + C$ which in the case $x = a$ becomes $= C$, if $x = a + dx$, will be

$$y = C + Pdx^m \pm Qdx^n;$$

since both of these values are either greater or smaller than C, depending on whether P was a positive or a negative quantity, in the case $x = a$ the function y will have either a minimum or a maximum value of the second kind [§ 278].

§349 Therefore, in such cases the true differentials of functions cannot be found by means of the usual rules for differentiation; those are only applicable, if the differential of the function is homogeneous to dx. But if in a singular case the differential of the function is expressed in terms of its power dx^n, then the rule yields 0 for this differential, if n was a number greater than 1; but on the other hand it exhibits an infinite times larger differential, if n is an exponent smaller than 1. So if the differential of $y = \sqrt{a-x}$ is in question in the case $x = a$, since $dy = -\frac{dx}{\sqrt{a-x}}$, having put $x = a$, $dy = -\frac{dx}{0}$ results. And if we wanted to derive the following differentials, because of the denominators $= 0$, all of them grow to infinity in the same way such that nothing can be concluded from this. But we saw that in this case $dy = \sqrt{-dx}$ and hence imaginary. But if one writes $x - dx$ instead of x, it will be $dy = \sqrt{dx}$ and hence it will be infinitely larger than dx such that dx vanishes with respect to dy. Hence, the usual rule even in this case does not cause any errors, since it exhibits the infinite value of dy.

§350 Therefore, one must not apply the usual rule, if in the series

$$pdx + \frac{1}{2}qdx^2 + \frac{1}{6}rdx^3 + \text{etc.},$$

expressing the complete differential of the function y, the first term p either becomes $= 0$ or infinite, but in this case the differential must be derived from the first principles. Therefore, if the differential corresponding to a given value of x of the function y, for which the letter p becomes either infinitely small or infinitely large, is in question, one has to go back to the first principles of differentiation. In all remaining cases in which neither $p = 0$ nor $p = \infty$ the usual rule will yield the true values of the the differential. Nevertheless, the case mentioned before (§ 348) is not to be neglected, if the function y contains a term of the form $(x-a)^n Q$ while n is fraction with an even denominator; for, even though one has lower differentials than Qdx^n, with respect to which this one vanishes, nevertheless, since Qdx^n, if dx is negative, becomes imaginary, this term Qdx^n transforms all remaining ones, with respect to which it vanishes, also into imaginary ones; this circumstance is especially to be considered in the case of curves. Therefore, I will explain some particular cases in which the true differential is not indicated by the common rule in the following examples.

EXAMPLE 1

Let the differential of the function $y = a + x - \sqrt{xx + ax - x\sqrt{2ax - xx}}$ be in question in the case $x = a$.

It is plain that the differential of this function $x = a$ is not found by means of the usual rule; for,

$$dy = dx + \frac{-xdx - \frac{1}{2}adx + \frac{1}{2}dx\sqrt{2ax-xx} + (axdx - xxdx) : \sqrt{2ax-xx}}{\sqrt{xx + ax - x\sqrt{2ax-xx}}};$$

for, having put $x = a$, it will be $dy = dx - \frac{adx}{a} = 0$. Therefore, let us start from the first principles of differentiation and at first, having written $x + dx$ instead of x, it will be

$$y^{\mathrm{I}} = a + x + dx$$
$$-\sqrt{xx + 2xdx + dx^2 + ax + adx - (x+dx)\sqrt{2ax - xx + 2adx - 2xdx - dx^2}}$$

But, having put $x = a$, it will be

$$y^{\mathrm{I}} = 2a + dx - \sqrt{2aa + 3adx + dx^2 - (a+dx)\sqrt{aa - dx^2}}.$$

Now, because $\sqrt{aa - dx^2} = a - \frac{dx^2}{2a}$ (for, the following terms can be neglected, since not all which are infinite times larger will be cancelled, as it will be seen soon), it will be

$$y^{\mathrm{I}} = 2a + dx - \sqrt{aa + 2adx + \frac{3}{2}dx^2}$$

and further, by extracting the root, it will be

$$y^{\mathrm{I}} = 2a + dx - \left(a + dx + \frac{dx^2}{4a}\right) = a - \frac{dx^2}{4a}.$$

But in the case $x = a$ it will be $y = a$; hence, because $y^{\mathrm{I}} = y + dy$, one will obtain

$$dy = -\frac{dx^2}{4a};$$

from this it is at the same time seen that the propounded function y becomes imaginary, if one puts $x = a$.

Example 2

To find the differential of this function $y = 2ax - xx + a\sqrt{aa - xx}$ in the case in which one puts $x = a$.

Having differentiated it in the usual way,

$$dy = 2adx - 2xdx - \frac{axdx}{\sqrt{aa - xx}},$$

which for $x = a$ goes over into infinity and hence is not indicated this way. But the differentials of the following orders in a like manner will become infinite such that not even from the series $pdx + \frac{1}{2}qdx^2 + \frac{1}{6}rdx^2 +$ etc. the true value of the differential can be found. Therefore, let us write $x + dx$ instead of x and we will have

$$y^{\mathrm{I}} = 2ax - xx + 2adx - 2xdx - dx^2 + a\sqrt{aa - xx - 2xdx - dx^2}$$

and, having put $x = a$, it will be

$$y^{\mathrm{I}} = aa - dx^2 + a\sqrt{-2adx - dx^2}.$$

But in the same case $y = aa$; therefore, it will be $dy = -dx^2 + a\sqrt{-2adx}$, and since dx^2 vanishes with respect to $\sqrt{-2adx}$, it will be

$$dy = a\sqrt{-2adx}.$$

Hence, if the differential dx is taken positively, dy will be imaginary; but if one writes $x - dx$ for x, it will be

$$dy = \sqrt{2adx};$$

because its value is a double value, the one positive, the other negative, the function y will have neither a maximum nor minimum value in the case $x = a$.

EXAMPLE 3

To find the differential of the function $y = 3aax - 3axx + x^3 + (a - x)\sqrt[3]{a^3 - x^3}$ *in the case* $x = a$.

Since this function is transformed into this form

$$y = a^3 - (a - x)^3 + (a - x)^{\frac{7}{3}}\sqrt[3]{aa + ax + xx},$$

having put $x = a + dx$,

$$y^{\mathrm{I}} = a^3 + dx^3 - dx^{\frac{7}{3}}\sqrt[3]{3aa},$$

and in the same case $y = a^3$. Therefore, it will be $dy = dx^3 - dx^{\frac{7}{3}} \sqrt[3]{3aa}$, and because dx^3 vanishes with respect to $dx^{\frac{7}{3}}$, it will be

$$dy = -dx^{\frac{7}{3}} \sqrt[3]{3aa};$$

therefore, in the case $x = a$ the function y has neither a maximum nor a minimum value.

EXAMPLE 4

To find the differential of the function $y + \sqrt{x} + \sqrt[4]{x^3} = (1 + \sqrt[4]{x})\sqrt{x}$ *in the case* $x = 0$.

Since the case $x = 0$ is propounded and in this case $y = 0$, write only dx instead of x and one will have

$$dy = dx^{\frac{1}{2}} + dx^{\frac{3}{2}} \quad \text{or} \quad dy = (1 + \sqrt[4]{dx})\sqrt{dx};$$

hence, at first it is plain that dx can not be taken negatively. But then, even though $\sqrt{dx}$ has a double value, the one positive, the other negative, nevertheless in this case, since its root $\sqrt[4]{dx}$ occurs, only the positive value can be taken. But $\sqrt[4]{dx}$ has both meanings and it will be

$$dy = \sqrt{dx} \pm \sqrt[4]{dx^3} \quad \text{and} \quad y^{\mathrm{I}} = 0 + \sqrt{dx} \pm \sqrt[4]{dx^3},$$

because of $y = 0$. Because both values of y^{I} are greater than y, it follows that in the case $x = 0$ y has a minimum value. But that the function $y = \sqrt{x} + \sqrt[4]{x^3}$ does not contain this one $-\sqrt{x} + \sqrt[4]{x^3}$, will become plain by reducing both to rational expressions. For, the first reduced this form $y - \sqrt{x} = \sqrt[4]{x}$ and squared gives $y^2 - 2y\sqrt{x} + x = x\sqrt{x}$ or $y^2 + x = (x + 2y)\sqrt{x}$ which squared again yields

$$y^4 - 2yyx - 4xxy + xx - x^3 = 0$$

The other $y + \sqrt{x} = \sqrt[4]{x^3}$ will give $y^2 + x = (x - 2y)\sqrt{x}$ and further

$$y^4 - 2yyx + 4xxy + xx - x^3 = 0,$$

which is different from the first. But on the other hand, the term $\sqrt[4]{x^3}$ retains the ambiguity of the sign. Therefore, this circumstance is to be considered in detail, that, even though in general the roots of even powers include both signs $+$ and $-$, nevertheless this ambiguity does not occur, if in the same expression higher roots of even powers of the same roots occur; these would be imaginary, if the first roots would be taken negatively. And from this source maxima and minima of the second kind follow, whenever such might not seem to occur.

EXAMPLE 5

To find the differential of the function

$$y = a + \sqrt{x-f} + (x-f)\sqrt[4]{x-f} + (x-f)^2\sqrt[8]{x-f}$$

in the case $x = f$.

Let us put $x - f = t$, and since $y = a + \sqrt{t} + t\sqrt[4]{t} + tt\sqrt[8]{t}$, the differential of this expression is in question in the case $t = 0$ in which $y = a$. Therefore, having written $t + dt$ or $0 + dt$ instead of t, it will be

$$y^{\mathrm{I}} = y + dy = a + \sqrt{dt} + dt\sqrt[4]{dt} + dt^2\sqrt[8]{dt}$$

and hence one will have

$$dy = \sqrt{dt} + dt\sqrt[4]{dt} + dt^2\sqrt[8]{dt}.$$

Here, at first it is plain that the differential can not be taken negatively, for, otherwise dy would become imaginary. But not only $\sqrt{dt}$, but even $\sqrt[4]{dt}$ can not be taken negatively; for, $\sqrt[8]{dt}$ would become imaginary; hence the differential dy has only the double value

$$dy = \sqrt{dt} + dt\sqrt[4]{dt} \pm dt^2\sqrt[8]{dt};$$

because both values are greater than zero, it follows that the function y has a minimum value of the second kind for $t = 0$ or $x = f$. Although in these cases the terms $dt\sqrt[4]{dt}$ and $dt^2\sqrt[8]{dt}$ vanish with respect to the first $\sqrt{dt}$, it is nevertheless to be taken into account, if the multiplicity of the values is considered, that the imaginary quantities are avoided.

EXAMPLE 6

To find the differential of the function $y = ax + bxx + (x-f)^n + (x-f)^{m+\frac{1}{2}}$ in the case $x = f$.

If one puts $x = f$, it will be $y = af + bff$, and if one writes $x + dx$ or $f + dx$ instead of x, the closest value will result as

$$y^{\mathrm{I}} = af + bff + adx + 2bfdx + bdx^2 + dx^n + dx^{m+\frac{1}{2}n},$$

such that

$$dy = adx + 2bfdx + bdx^2 + dx^n + dx^m\sqrt{dx^n}.$$

Therefore, if n is not an even number, the differential dx cannot be taken negatively. But the last term $dx^m\sqrt{dx^n}$ has an ambiguous sign; therefore, y^{I} will have two values, both greater than the one of y, if $a + 2bf$ was a positive quantity and the exponents n and $m + \frac{1}{2}n$ were greater than 1. Therefore, the value of the function y in the case $x = f$ will be a minimum and this happens, whether n is an integer number or a fraction, as long as only the numerator was not also an even number in that case.

§351 But this method to deduce differentials from the first principles is especially useful in the case of transcendental functions, since in certain cases the differential found in the usual way either vanishes or seems to grow to infinity. But here species of the infinite and the infinitely small appear which are not found in algebraic functions. For, because, if i denotes an infinite number, $\log i$ is also infinite, but nevertheless has an infinitely small ratio to the number i and even to any power i^n, no matter how small the exponent n is set, the fraction $\frac{\log i}{i^n}$ will be infinitely small and can not be finite, unless the exponent n becomes infinitely small. Therefore, $\log i$ will be homogeneous to i^n, if the exponent n was infinitely small. Now let us put $i = \frac{1}{\omega}$ while ω is an infinitely small quantity; $-\log\omega$ will be homogeneous to $\frac{1}{\omega^n}$, if the exponent n is infinitely small, and hence $-\frac{1}{\log\omega}$ will be homogeneous to ω^n; and hence, $-\frac{1}{\log dx}$ will be infinitely small compared to dx^n, while n is an infinitely small fraction. So, if it was $y = -\frac{1}{\log x}$, the differential of y in the case $x = 0$ will be $= -\frac{1}{\log dx} = dx^n$ and hence dy will have an infinite ratio to dx and to any power of dx; and with respect to $-\frac{1}{\log dx}$ completely all powers of dx vanish, no matter how small their exponents were.

§352 Further, we also saw, if a was a number greater than 1 and i was infinite, that a^i will be infinite of such a high degree that with respect to it not only i but also any power of i vanishes; and i^n does not become homogeneous to

a^i until the exponent n was increased to infinity. Now, let $i = \frac{1}{\omega}$ such that ω denotes the infinitely small; $a^{\frac{1}{\omega}}$ will be homogeneous to $\frac{1}{\omega^n}$ while n denotes an infinitely large number and hence $a^{\frac{-1}{\omega}}$ or $\frac{1}{a^{1:\omega}}$ will be infinitely large compared to ω^n. Therefore, $\frac{1}{a^{1:dx}}$ will be infinitely small, but vanishes with respect to all powers of dx, because it is homogeneous to the power dx^n, while n is an infinitely large number. Hence, if the differential of $y = \frac{1}{a^{1:x}}$ is in question in the case $x = 0$, since $y = 0$, it will be $dy = \frac{1}{a^{1:dx}}$ and hence is infinite times smaller than each power of dx.

§353 But if a was a number smaller than 1, then, because $\frac{1}{a}$ becomes larger than 1, the question is reduced to the preceding case. If one has the expression $a^{\frac{1}{\omega}}$, by putting $a = 1 : b$ it will be transformed into $b^{-\frac{1}{\omega}}$ or $\frac{1}{b^{1:\omega}}$ which, because of $b > 1$, will be homogeneous to ω^n while n denotes any infinitely large number. Therefore, having mentioned these things in advance, we will be able to resolve the following examples.

EXAMPLE 1

To find the differential of the function $y = xx - \frac{1}{\log x}$ in the case $x = 0$.

Since for $x = 0$ also $y = 0$, if we write $x + dx$ or $0 + dx$ instead of x, it will be

$$y^{\mathrm{I}} = dy = dx^2 - \frac{1}{\log dx}.$$

But because $-\frac{1}{\log dx}$ is homogeneous to dx^n while n denotes an infinitely small number, with respect to it dx^2 will vanish and it will be

$$dy = -\frac{1}{\log dx} = dx^n.$$

But because the logarithms of negative numbers are imaginary, dx cannot be taken negatively and therefore in the case $x = 0$ the function y will have a minimum value, but a minimum extending neither to the first nor the second kind. It certainly does not extend to the first kind, since y does not have any preceding very close values, but is only smaller than the following values, if x is greater than zero. But it also does not extend to second kind, since the following values, it is compared to, are not double values; and therefore a third kind of maxima and minima results that only occurs in logarithmic and transcendental functions, but never occurs in algebraic functions; this will be treated in the following book on curves in more detail.

EXAMPLE 2

To find the differential of the function $y = (a-x)^n - x^n(\log a - \log x)^n$ *in the case* $x = a$.

This differential, if n is not an integer number, can be found from the general formula

$$dy = pdx + \frac{1}{2}qdx^2 + \frac{1}{6}rdx^3 + \text{etc.};$$

for, it will be

$$pdx = -n(a-x)^{n-1}dx - nx^{n-1}dx(\log a - \log x)^n + nx^{n-1}(\log a - \log x)^{n-1}dx,$$

which value vanishes for $x = a$; for, even if $n = 1$, it will be

$$pdx = -dx + dx = 0.$$

Therefore, if we proceed further, it will be

$$\frac{1}{2}qdx^2 = \frac{n(n-1)}{1\cdot 2}(a-x)^{n-2}dx^2 - \frac{n(n-1)}{1\cdot 2}x^{n-1}dx^2(\log a - \log x)^n + \frac{n^2}{2}x^{n-2}dx^2(\log a - \log x)^{n-1}$$
$$+\frac{n(n-1)}{1\cdot 2}x^{n-2}dx^2(\log a - \log x)^{n-1} - \frac{n(n-1)}{1\cdot 2}dx^2(\log a - \log x)^{n-2}.$$

Hence, if it was $n = 1$, it will be $\frac{1}{2}qdx^2 = \frac{dx^2}{2a}$ for $x = a$. In like manner, if $n = 2$, one would have to proceed up to the term $\frac{1}{6}rdx^3$ and so fourth. Therefore, one will more conveniently use the principles of differentiation, and because for $x = a$ $y = 0$, if we write $x + dx$ or $a + dx$ instead of x, it will be

$$y^{\mathrm{I}} = (-dx)^n - (a+dx)^n(\log a - \log(a+dx))^n = y + dy = dy,$$

because of $y = 0$. But

$$\log(a+dx) = \log a + \frac{dx}{a} - \frac{dx^2}{2a^2} + \frac{dx^3}{3a^3} - \text{etc.},$$

whence

$$dy = (-dx)^n - \left(a^n + na^{n-1}dx + \frac{n(n-1)}{1\cdot 2}a^{n-2}dx^2 + \text{etc.}\right)\left(-\frac{dx}{a} + \frac{dx^2}{2a^2} - \frac{dx^3}{3a^3} + \text{etc.}\right)^n$$

$$= \frac{n}{2a}(-dx)^{n+1}.$$

Therefore, in the case $x = a$ the differential dy in question of the propounded formula will be as follows:

$$\begin{array}{ll|lll}
\text{if} & n=1 & dy & = & \dfrac{dx^2}{2a}, \quad \text{as we found before} \\
\text{if} & n=2 & dy & = - & \dfrac{2dx^3}{2a} \\
\text{if} & n=3 & dy & = & \dfrac{3dx^4}{2a} \\
\text{if} & n=4 & dy & = - & \dfrac{4dx^5}{2a} \\
\text{etc.} & & \text{etc.} & &
\end{array}$$

Therefore, if n was an odd number, in the case $x = a$ the function has a minimum value, but if n is an even number, neither a maximum nor a minimum value; the same holds, if n was a fraction with an odd denominator. But if n was a fraction with an even denominator, dx has to be taken negatively, so that we do not get to imaginary quantities; and because of the ambiguous meaning, the function will also have neither a maximum value nor a minimum value.

EXAMPLE 3

To find the differential of the function $y = x^x$ in the case $x = \frac{1}{e}$, where e denotes the number whose hyperbolic logarithm is $= 1$.

Since in general $dy = x^x dx(\log x + 1)$, this differential vanishes in the case $x = \frac{1}{e}$ or $\log x = -1$. Therefore, compare this differential to the general form $pdx + \frac{1}{2}qdx^2 + \text{etc.}$; it will be

$$p = x^x(\log x + 1) \quad \text{and} \quad q = x^x(\log x + 1)^2 + x^{x-1}$$

and, having put $\log x = -1$ or $x = \frac{1}{e}$, it will be

$$q = \left(\frac{1}{e}\right)^{\frac{1-e}{e}} = e^{\frac{e-1}{e}}.$$

Therefore, the differential in question will be

$$dy = \frac{1}{2} e^{(e-1):e} dx^2$$

and therefore the function $y = x^x$ will have a minimum value in the case $x = \frac{1}{e}$.

EXAMPLE 4

To find the differential of this function $y = x^n + e^{-1:x}$ in the case in $x = 0$.

Since for $x = 0$ also $y = 0$, if one puts $x = 0 + dx$, it will be

$$y^{\mathrm{I}} = dy = dx^n + \frac{1}{e^{1:dx}}.$$

But we saw that $\frac{1}{e^{1:x}}$ is homogeneous to the infinite power of dx or to dx^∞ and will hence vanish with respect to dx^n such that

$$dy = dx^n.$$

§354 What happens in the first differentials in certain cases, i.e. that they are not found by means of the usual rules of differentiation, also happens in differentials of the second and third and higher order in the cases in which in the complete differential form

$$d.y = pdx + \frac{1}{2}qdx^2 + \frac{1}{6}rdx^3 + \frac{1}{24}sdx^4 + \text{etc.}$$

some of the quantities q, r, s etc. either vanish or become infinite. Because [§ 339]

$$dd.y = qdx^2 + rdx^3 + \frac{7}{12}sdx^4 + \text{etc.},$$

if in which case $q = 0$, it will be $ddy = rdx^3$; but if in the same case also r vanishes, from this series the second differential can not be found at all, but one will have to go back to the principles of differentials; writing $x + dx$

instead of x find the value y^{I} and by writing $x + 2dx$ instead of x find the value of y^{II}, having done which the true value of the second differential will be

$$ddy = dy^{\mathrm{I}} - dy = y^{\mathrm{II}} - 2y^{\mathrm{I}} + y.$$

If in like matter if the the third differential is in question, additionally, write $x + 3dx$ instead of x in y and, having found the value y^{III}, it will be

$$d^3y = y^{\mathrm{III}} - 2y^{\mathrm{II}} + 3y^{\mathrm{I}} - y$$

and so fourth. We will illustrate these cases in the following examples.

EXAMPLE 1

To find the second differential of the function $y = \frac{aa-xx}{aa+xx}$ in the case $x = \frac{a}{\sqrt{3}}$.

Investigating the complete differential of y from the form

$$dy = pdx + \frac{1}{2}qdx^2 + \frac{1}{6}rdx^3 + \frac{1}{24}sdx^4 + \text{etc.}$$

the following values will result for p, q, r, s etc.

$$p = -\frac{4aax}{(aa+xx)^2}, \quad q = \frac{-4a^4 + 12aaxx}{(aa+xx)^3} \quad \text{and} \quad r = \frac{48a^4x - 48aax^3}{(aa+xx)^4}.$$

Since now

$$ddy = rdx^3 + \frac{7}{12}sdx^4 + \text{etc.},$$

because of $q = 0$, in the case $x = \frac{a}{\sqrt{3}}$ and in the same case $r = \frac{27\sqrt{3}}{8a^3}$, the second differential in question will be

$$ddy = \frac{27dx^2\sqrt{3}}{8a^3}.$$

EXAMPLE 2

To find the third differential of the function $y = \frac{aa-xx}{aa+xx}$ in the case $x = a$.

As before by finding the complete differential

$$dy = \frac{1}{2}qdx^2 + \frac{1}{6}rdx^3 + \frac{1}{24}sdx^4 + \text{etc.},$$

since the third differential is $d^3y = rdx^3 + \frac{3}{2}sdx^4 + \text{etc.}$, because of

$$r = \frac{48a^4x - 48aax^3}{(aa+xx)^4},$$

it will be $x = a$ in the case $r = 0$; hence, one has to proceed to the value s which will be

$$s = \frac{48a^4 - 144aaxx}{(aa+xx)^4} - \frac{8x(48a^4x - 48aax^3)}{(aa+xx)^5};$$

therefore, having put $x = a$, it will be $s = -\frac{96a^4}{2^4a^8} = -\frac{6}{a^4}$; therefore, in this case it will be

$$d^3y = -\frac{9dx^4}{a^4}.$$

EXAMPLE 3

To find the differential of arbitrary order of the function $y = ax^m + bx^n$ in the case $x = 0$.

By successively writing $x + dx$, $x + 2dx$, $x + 3dx$ etc. instead of x the following values of the function y will be

$$\begin{aligned} y^{\mathrm{I}} &= a(x+dx)^m + b(x+dx)^n, \\ y^{\mathrm{II}} &= a(x+2dx)^m + b(x+2dx)^n, \\ y^{\mathrm{III}} &= a(x+3dx)^m + b(x+3dx)^n, \\ &\text{etc.} \end{aligned}$$

Therefore, having put $x = 0$, it will be $y = 0$ and its differentials will be

$$
\begin{aligned}
dy &= adx^m + bdx^n,\\
ddy &= (2^m - 1)adx^m + (2^n - 2)bdx^n,\\
d^3y &= (3^m - 3\cdot 2^m + 3)adx^m + (3^n - 3\cdot 2^n + 3)bdx^n,\\
d^4y &= (4^m - 4\cdot 3^m + 6\cdot 2^m - 4)adx^m + (4^n - 4\cdot 3^n + 6\cdot 2^n - 4)bdx^n\\
&\quad \text{etc.}
\end{aligned}
$$

Therefore, if the exponent n was greater than m, the second terms in these expressions vanish with respect to the first. Nevertheless, it is to be considered, if n was a fractional number, that the cases in which these differentials become either imaginary or ambiguous can be distinguished. It will be convenient reserve the further discussion of these cases for the doctrine of curves.